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A NOTE ON STABILIZING DAMPING CONFIGURATIONS FOR LINEAR NONCONSERVATIVE SYSTEMS

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Abstract—It is well known that a linear elastic system having a stable equilibrium for zero damping may be destabilized by the introduction of nonzero damping. In this note a class of damping configurations not having this destabilizing effect is determined.

INTRODUCTION

LET US consider a linear n-degree-of-freedom system described by

$$\mathbf{M}\ddot{\mathbf{y}}+\mathbf{C}\dot{\mathbf{y}}+\mathbf{K}\mathbf{y} = 0 \tag{1}$$

where $\mathbf{y}$ is a real n-vector and $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$ are real $n \times n$ matrices with $\mathbf{M}$ and $\mathbf{K}$ nonsingular. We make no assumptions regarding symmetry or positivity of these matrices. Many studies have shown that even if the equilibrium of (1) is stable for $\mathbf{C} = 0$, it may become unstable for $\mathbf{C} = \varepsilon\mathbf{C}_0$ when $\mathbf{C}_0$ is some given matrix and ε is an arbitrarily small positive number [1–5]. It has recently been shown that one form of $\mathbf{C}$ which does not have this destabilizing effect is given by $\mathbf{C} = \varepsilon\mathbf{M}$ where ε is a positive real number [6].

The purpose of this note is to enlarge considerably the class of matrices C which are "nondestabilizing" under the assumption that the equilibrium of (1) is stable for $C = 0$.

RESULTS AND PROOFS

Our first result is given by the following theorem:

Theorem 1. If the equilibrium of (1) is stable for $\mathbf{C} = 0$, it is also stable for any matrix $\mathbf{C}$ such that

$$\mathbf{C} = \sum_{p=-\infty}^{\infty} c_p\mathbf{M}(\mathbf{M}^{-1}\mathbf{K})^p, \qquad c_p \geq 0. \tag{2}$$

If $\mathbf{C}$ is also nonsingular, the equilibrium of (1) is asymptotically stable.

We note that the class of matrices defined by (2) includes the case of "proportional damping",

$$\mathbf{C} = c_o\mathbf{M}+c_1\mathbf{K}, \qquad c_o \geq 0, \qquad c_1 \geq 0, \tag{3}$$

as well as such esoteric forms as

$$\mathbf{C} = \mathbf{M}\exp(\mathbf{M}^{-1}\mathbf{K}). \tag{4}$$

The result of [6] is obtained from Theorem 1 if $c_p = 0$ for all $p \neq 0$ and $c_o > 0$.

Theorem 1 can be quite useful from the standpoint of synthesizing a matrix **C** which does not destabilize (1), but it may not be too useful from the standpoint of analysis. That is, given a matrix **C** it may be quite difficult to determine whether or not **C** can be decomposed in the form (2). This problem motivates our second result:

Theorem 2. If the equilibrium of (1) is stable for $\mathbf{C} = 0$, it is asymptotically stable for any matrix **C** such that

$$\mathbf{C} = \mathbf{C}_1 + \varepsilon \mathbf{C}_2 \tag{5}$$

$$\mathbf{C}_1 = \sum_{p=-\infty}^{\infty} c_p \mathbf{M}(\mathbf{M}^{-1}\mathbf{K})^p, \qquad c_p \geq 0, \tag{6}$$

provided $\mathbf{C}_1$ is nonsingular and $|\varepsilon|$ is sufficiently small.

Before proving these theorems, let us first establish the following lemma:

Lemma 1. If the equilibrium of (1) is stable for $\mathbf{C} = 0$, there exists a real $n \times n$ matrix **G** such that both **G** and $\mathbf{GM}^{-1}\mathbf{K}$ are positive definite and symmetric.

We will give a constructive proof of Lemma 1. Let us first note that the equilibrium of (1) is stable for $\mathbf{C} = \mathbf{0}$ if and only if there exist n linearly independent real n-vectors f_i such that

$$\mathbf{M}^{-1}\mathbf{K} f_i = \lambda_i f_i \tag{7}$$

for some positive real numbers λ_i. Thus there must also exist n linearly independent real n-vectors g_i such that

$$(\mathbf{M}^{-1}\mathbf{K})^T g_i = \lambda_i g_i, \tag{8}$$

implying

$$g_i^T \mathbf{M}^{-1}\mathbf{K} = \lambda_i g_i^T, \tag{9}$$

where $(\,.\,)^T$ denotes the transpose of $(\,.\,)$. We now consider the matrix

$$\mathbf{G} = \sum_{i=1}^{n} \alpha_i g_i g_i^T \tag{10}$$

where the α_i are arbitrary positive real numbers. We note that **G** is a real symmetric positive definite matrix since the vectors g_i, $i = 1, 2, \ldots, n$, are real and linearly independent. We also find by (9) that $\mathbf{GM}^{-1}\mathbf{K}$,

$$\mathbf{GM}^{-1}\mathbf{K} = \sum_{i=1}^{n} \alpha_i \lambda_i g_i g_i^T \tag{11}$$

is a real symmetric positive definite matrix since the λ_i are positive real numbers.

We now recall two theorems previously proved elsewhere [7]:

Theorem 3. If there exists a matrix **G** such that $\mathbf{GM}^{-1}\mathbf{C}$ is nonnegative, while **G** and $\mathbf{GM}^{-1}\mathbf{K}$ are symmetric and positive definite, the equilibrium of (1) is stable.

Theorem 4. If there exists a matrix **G** such that $\mathbf{GM}^{-1}\mathbf{C}$ is positive definite, while **G** and $\mathbf{GM}^{-1}\mathbf{K}$ are symmetric and positive definite, the equilibrium of (1) is asymptotically stable.

Now let us consider the matrices **C** and **G** given by (2) and Lemma 1. We find

$$\begin{aligned}\mathbf{GM}^{-1}\mathbf{C} &= \mathbf{G}\left(\sum_{p=-\infty}^{+\infty} c_p(\mathbf{M}^{-1}\mathbf{K})^p\right) \\ &= \sum_{p\,\text{even}} c_p[(\mathbf{M}^{-1}\mathbf{K})^{p/2}]^T\mathbf{G}[(\mathbf{M}^{-1}\mathbf{K})^{p/2}] \\ &\quad + \sum_{p\,\text{odd}} c_p[(\mathbf{M}^{-1}\mathbf{K})^{(p-1)/2}]^T\mathbf{GM}^{-1}\mathbf{K}[(\mathbf{M}^{-1}\mathbf{K})^{(p-1)/2}].\end{aligned} \tag{12}$$

We see that $\mathbf{GM}^{-1}\mathbf{C}$ is symmetric and nonnegative if all c_p are nonnegative since **G** and $\mathbf{GM}^{-1}\mathbf{K}$ are symmetric and positive definite. If **C** is nonsingular, so is $\mathbf{GM}^{-1}\mathbf{C}$. Therefore $\mathbf{GM}^{-1}\mathbf{C}$ is positive definite if all c_p are nonnegative and **C** is nonsingular. Thus by Theorems 3 and 4 we have established Theorem 1.

Theorem 2 is now easily established by noting that $\mathbf{GM}^{-1}\mathbf{C}_1$ is positive definite and $\mathbf{GM}^{-1}(\mathbf{C}_1+\varepsilon\mathbf{C}_2)$ is continuously dependent upon ε. Since $\mathbf{GM}^{-1}(\mathbf{C}_1+\varepsilon\mathbf{C}_2)+(\mathbf{GM}^{-1}(\mathbf{C}_1+\varepsilon\mathbf{C}_2))^T$ is positive definite for $\varepsilon = 0$, it remains positive definite for $|\varepsilon|$ sufficiently small. Thus $\mathbf{GM}^{-1}\mathbf{C}$ is positive definite if $|\varepsilon|$ is sufficiently small and by Theorems 3 and 4 we have proven Theorem 2.

CONCLUSIONS

Theorems 1 and 2 considerably enlarge the class of damping configurations which assuredly do not destabilize a linear nonconservative system. Theorem 1 would seem to be most useful in synthesizing such damping configurations, while Theorem 2 may be more useful for the analysis of a given damping configuration.

It should be noted that Theorems 1 and 2 provide sufficient but not necessary conditions for **C** to be "nondestabilizing". A condition which comes much closer to being necessary, as well as sufficient, is that there exists a matrix **G** satisfying Theorem 3. However this lacks the simplicity of the conditions in Theorems 1 and 2.

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Абстракт—Хорошо известно, что линейная, упругая система, имеющая устойчивое равновесие для нулевого демпфирования может потерять устойчивость путем введения ненулевого демпфирования. В настоящей заметке определяется класс очертаний демпфирования, не имеющих этого дестабилизированного эффекта.